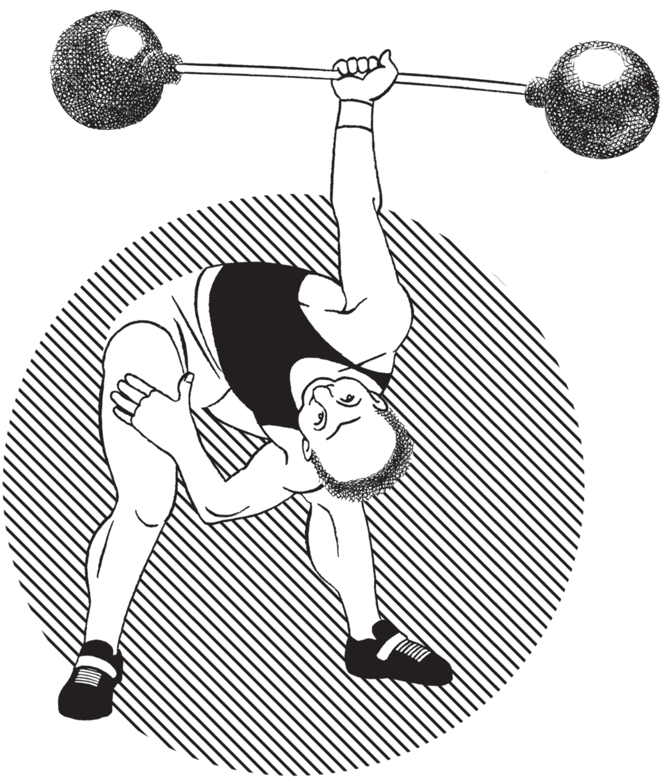
Department of Mathematics and Systems Analysis

Weight theory on bounded domains and metric measure spaces

Emma-Karoliina Kurki



Aalto University

DOCTORAL THESES

Aalto University publication series
DOCTORAL THESES 151/2022

Weight theory on bounded domains and metric measure spaces

Emma-Karoliina Kurki

A doctoral thesis completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall M1 of the school on 25 November 2022 at 12 noon.

Aalto University
School of Science
Department of Mathematics and Systems Analysis
Nonlinear Partial Differential Equations Research Group

Supervising professor
Prof. Juha Kinnunen, Aalto University

Thesis advisor
Prof. Juha Kinnunen, Aalto University

Preliminary examiners
Research Fellow Amiran Gogatishvili, Czech Academy of Sciences, Czechia
Prof. Sibei Yang, Lanzhou University, P. R. China

Opponent
Prof. Sheldy Ombrosi, Universidad Nacional del Sur, Argentina

Aalto University publication series
DOCTORAL THESES 151/2022

ISBN 978-952-64-0985-6 (printed)
ISBN 978-952-64-0986-3 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-64-0986-3

Unigrafia Oy
Helsinki 2022

Finland

Printed matter
4041-0619

A! Aalto University

Abstract

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

Author
Emma-Karoliina Kurki

Name of the doctoral thesis
Weight theory on bounded domains and metric measure spaces

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL THESES 151/2022

Field of research Mathematics

Manuscript submitted 8 August 2022 **Date of the defence** 25 November 2022

Permission for public defence granted (date) 22 September 2022 **Language** English

☐ **Monograph** ☒ **Article thesis** ☐ **Essay thesis**

Abstract

A weight is a nonnegative, locally integrable function. Muckenhoupt weights are a prominent class of weights in the study of harmonic analysis and partial differential equations. The present thesis contributes to the theory of local weights defined on a bounded Euclidean domain, as well as weights on metric measure spaces with a doubling measure.

We show a two-weight Sobolev-Poincaré inequality on a Boman domain by the dyadic sparse domination method. We first obtain a local weighted inequality for an integral operator supported on a subcollection of dyadic cubes and majoring a continuous operator pointwise. The local inequality is then propagated by a chaining argument. As an application we obtain Poincaré inequalities for certain powers of distance functions, and supersolutions of the p-Laplace equation.

A theorem by Wolff states that a weight defined on a measurable subset and satisfying a Muckenhoupt-type compatibility condition has an extension into the whole space. We generalize this theorem to metric measure spaces with a doubling measure. Related to the extension problem, we obtain estimates for Muckenhoupt weights on Whitney chains.

We give 11 different characterizations for functions satisfying a weak reverse Hölder inequality. Most importantly, we show that the weak reverse Hölder and weak A-infinity conditions are equivalent in metric spaces with a doubling measure. This is not true of the classical reverse Hölder and A-infinity conditions, unless the measure satisfies another regularity property such as annular decay.

The natural maximal and minimal functions commute pointwise with the logarithm of a Muckenhoupt weight. We use this observation to characterize the limiting cases of Muckenhoupt and reverse Hölder conditions. The characterization yields a simple proof of a refined Jones factorization theorem. In addition, we show a boundedness result for the natural maximal function.

Keywords Muckenhoupt weight, weighted Poincaré inequality, metric measure space, doubling condition, extension, reverse Hölder inequality, maximal function

ISBN (printed) 978-952-64-0985-6 **ISBN (pdf)** 978-952-64-0986-3

ISSN (printed) 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Helsinki **Location of printing** Helsinki **Year** 2022

Pages 142 **urn** http://urn.fi/URN:ISBN:978-952-64-0986-3

Aalto-yliopisto

Aalto-yliopisto, PL 11000, 00076 Aalto www.aalto.fi

Tiivistelmä

Tekijä
Emma-Karoliina Kurki

Väitöskirjan nimi
Painojen teoriaa rajoitetussa alueessa ja metrisessä mitta-avaruudessa

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

Sarja Aalto University publication series DOCTORAL THESES 151/2022

Tutkimusala Matematiikka

Käsikirjoituksen pvm 08.08.2022 **Väitöspäivä** 25.11.2022

Väittelyluvan myöntämispäivä 22.09.2022 **Kieli** Englanti

☐ **Monografia** ☒ **Artikkeliväitöskirja** ☐ **Esseeväitöskirja**

Tiivistelmä

Paino on epänegatiivinen, lokaalisti integroituva funktio. Muckenhouptin painot ovat harmonisessa analyysissä ja osittaisdifferentiaaliyhtälöiden teoriassa tärkeä painofunktioiden luokka. Väitöskirja käsittelee Muckenhouptin painojen teoriaa yhtäältä euklidisen avaruuden rajoitetuissa alueissa ja toisaalta metrisissä mitta-avaruuksissa, joissa mitta on tuplaava.

Euklidisen avaruuden Boman-alueessa todistetaan kahden painon Sobolev-Poincaré-epäyhtälö käyttäen dyadisia tekniikoita. Alkuperäistä jatkuvaa integraalioperaattoria rajoitetaan pisteittäin dyadisella operaattorilla, jonka kantaja sisältyy valikoimaan dyadisia alikuutioita. Dyadiselle operaattorille todistetaan painotettu normiepäyhtälö, joka yleistyy koko alueeseen Whitney-ketjuja pitkin. Sovelluksina näytetään Poincarén epäyhtälöt eräille etäisyysfunktion potensseille ja p-Laplace-yhtälön superratkaisuille.

Wolffin jatkolause euklidisissa avaruuksissa väittää, että avaruuden mitallisessa osajoukossa määritelty Muckenhouptin paino vodaan jatkaa koko avaruuteen, jos ja vain jos se toteuttaa Muckenhoupt-tyyppisen yhteensopivuusehdon. Jatkolause yleistetään metrisiin mitta-avaruuksiin, joissa mitta on tuplaava. Jatko-ongelmaan liittyen näytetään estimaatteja Muckenhouptin painoille Whitney-ketjuilla.

Heikko käänteinen Hölderin epäyhtälö on käänteisen Hölderin epäyhtälön ei-tuplaava yleistys. Väitöskirjaan sisältyvässä artikkelissa annetaan kaikkiaan 11 karakterisaatiota funktioille, jotka toteuttavat heikon käänteisen Hölderin epäyhtälön. Erityisesti todistetaan, että tämä epäyhtälö on yhtäpitävä heikon Muckenhouptin ehdon kanssa, kunhan avaruuden mitta on tuplaava.

Luonnollinen maksimaali- ja minimaalifunktio kommutoivat Muckenhouptin painon logaritmin kanssa. Tätä ominaisuutta hyödyntäen karakterisoidaan käänteisen Hölderin ja Muckenhouptin ehtojen rajatapaukset. Lisäksi näytetään rajoitettuustulos luonnolliselle maksimaalifunktiolle.

Avainsanat Muckenhouptin paino, painotettu Poincarén epäyhtälö, metrinen mitta-avaruus, tuplaavuusehto, funktion jatkaminen, käänteinen Hölderin epäyhtälö, maksimaalifunktio

ISBN (painettu) 978-952-64-0985-6		**ISBN (pdf)** 978-952-64-0986-3	
ISSN (painettu) 1799-4934		**ISSN (pdf)** 1799-4942	
Julkaisupaikka Helsinki	**Painopaikka** Helsinki	**Vuosi** 2022	
Sivumäärä 142	**urn** http://urn.fi/URN:ISBN:978-952-64-0986-3		

Preface

The research work for this thesis was carried out at the Department of Mathematics and Systems Analysis at Aalto University between May 2018 and July 2022. My sincerest thanks belong to those who have accompanied me in the effort in one way or another: My advisor-supervisor Juha Kinnunen, without whom this work would have neither begun nor ended. My coauthors Carlos Mudarra and Antti V. Vähäkangas. The Emil Aaltonen and Vilho, Yrjö, and Kalle Väisälä Foundations for their very generous financial support. Hauenkuonon Igni Ferroque and Yes Yes No Maybe helped with the really tough bits. Family, friends, and coworkers, with special thanks to Mili for inspiring the cover illustration. The coffee room gang, for the beautiful moments and girlfriends we shared.

August 8, 2022,

Emma-Karoliina Kurki

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Kurki, E.-K. and Vähäkangas, A. V. Weighted norm inequalities in a bounded domain by the sparse domination method. *Rev. Mat. Complut.*, 34, no. 2, 435–467, 2021.

II Kurki, E.-K. and Mudarra, C. On the extension of Muckenhoupt weights in metric spaces. *Nonlinear Anal.*, 215, paper no. 112671, 2022.

III Kinnunen, J., Kurki, E.-K., and Mudarra, C. Characterizations of weak reverse Hölder inequalities on metric measure spaces. *Math. Z.*, 301, no. 3, 2269—2290, 2022.

IV Kurki, E.-K. Limiting conditions of Muckenhoupt and reverse Hölder classes on metric measure spaces. *Available at arXiv:2204.01441*, April 2022.

Author's Contribution

Publication I: "Weighted norm inequalities in a bounded domain by the sparse domination method"

A. V. Vähäkangas proved the main result of the paper. E.-K. Kurki and A. V. Vähäkangas wrote the paper up.

Publication II: "On the extension of Muckenhoupt weights in metric spaces"

Both authors contributed equally to all parts of the paper.

Publication III: "Characterizations of weak reverse Hölder inequalities on metric measure spaces"

J. Kinnunen proposed the problem and suggested techniques. E.-K. Kurki and C. Mudarra conducted the major part of the research, contributing equally to all parts of the paper.

Publication IV: "Limiting conditions of Muckenhoupt and reverse Hölder classes on metric measure spaces"

The paper represents independent work by the author.

1. Introduction

Muckenhoupt, or A_p, weights are an important class of weights at the heart of this study. For $1 < p < \infty$, $A_p(X)$ weights are precisely those nonnegative, locally integrable functions w that satisfy

$$\sup_{B \subset X} \left(\fint_B w \, \mathrm{d}\mu \right) \left(\fint_B w^{-\frac{1}{p-1}} \, \mathrm{d}\mu \right)^{p-1} < \infty. \tag{1.1}$$

This supremum is denoted $[w]_p$ and called the characteristic A_p constant of w. When $p = 1$, we require instead that there exist a constant $[w]_1 < \infty$ such that for every ball $B \subset X$

$$\fint_B w \, \mathrm{d}\mu \le [w]_1 \operatorname*{ess\,inf}_{x \in B} w(x). \tag{1.2}$$

The present thesis contributes to two neglected aspects of the theory of Muckenhoupt weights, namely local weights and weights on metric measure spaces. In Chapter 2, based on Publication I, we work in Euclidean spaces with weights defined on a bounded domain $\Omega \subset \mathbb{R}^n$ instead of the whole space $X = \mathbb{R}^n$. Here we show a two-weight Sobolev-Poincaré inequality based on local data by the dyadic sparse domination method.

Chapter 3 and Publications II–IV focus on metric spaces supporting a doubling measure. The principal results are an extension theorem for Muckenhoupt weights, characterizing weights that satisfy a weak reverse Hölder inequality, and characterizing the limiting classes of A_p and reverse Hölder classes of functions. While most if not all of these results are generalizations of their Euclidean counterparts, in metric measure spaces the structure is laid bare, revealing phenomena not seen in $\mathbb{R}^n$. For us chief among these is the fact that weights satisfying a reverse Hölder inequality are no longer necessarily Muckenhoupt. This further raises the question of determining those structural conditions on which the Muckenhoupt–reverse Hölder equivalence depends.

1.1 Meeting Muckenhoupt weights

Muckenhoupt's concern in introducing the eponymous weights in his 1972 article [51] was the mean summability of Fourier series, which is achieved in weighted norm with the weight satisfying (1.1)–(1.2). The following year, Gehring [30] discovered that the Jacobian of a quasiconformal mapping satisfies a reverse Hölder inequality (and hence is a Muckenhoupt weight, although Gehring does not make the connection; see also Chapter 3.3 below), leading to a higher integrability result. Elcrat–Meyers [50] likewise applied reverse Hölder inequalities to obtain higher integrability results in calculus of variations. The body of research on weights and weighted inequalities grew rapidly throughout the 1970s, as demonstrated by [25, 52].

The basic problem in the study of weighted norm inequalities is to obtain estimates of the form

$$v\left(\{x \in X : |Tf(x)| > \lambda\}\right) \leq \frac{C}{\lambda^p} \int_X |f|^p \, w \, \mathrm{d}x \,, \quad \text{or} \tag{1.3}$$

$$\int_X |Tf|^p \, v \, \mathrm{d}x \leq C \int_X |f|^p \, w \, \mathrm{d}x \,, \tag{1.4}$$

where $1 \leq p < \infty$, and T is an operator such as a maximal function or singular integral. The weight functions v and w may be the same, i.e. $v = w$, or different, setting apart one-weight and two-weight inequalities. It turns out that if we take $v = w$ and T to be the Hardy–Littlewood maximal operator, or indeed any singular integral with a sufficiently smooth kernel [14], the A_p condition (1.1)–(1.2) (1.1)–(1.2) is necessary and sufficient to establish the inequality (1.3) whenever $p = 1$, and (1.4) whenever $p > 1$. As for the two-weight case, several fundamental problems remain open, including identifying the analogy of the A_p condition. We refer to [18] for a survey and many more bibliographic references.

Of particular interest to us are weighted Poincaré inequalities, that will be discussed in Chapter 2. Consider nonlinear partial differential equations of divergence type

$$\operatorname{div} A(x, u, \mathrm{D}u) = 0.$$

The regularity of the solution depends on the structural properties of the differential operator A. In the early article by Fabes, Kenig, and Serapioni [26], the authors are interested in relaxing one of these, namely the ellipticity of the operator $A(x, \mathrm{D}u) = A(x)\mathrm{D}u(x)$. To this end, they replace the ellipticity condition with a weighted version

$$Cw\,|\xi|^2 \leq A\xi \cdot \xi \leq C_1 w\,|\xi|^2 \,,$$

where the nonnegative weight w is allowed to either vanish, be infinite, or both. (In the classical situation $w \equiv 1$.) One might ask what are the minimal structural conditions which guarantee that the solution has a given amount of regularity, such as local Hölder continuity. In particular, what are the conditions on the weight w? Fabes et al established a weighted Poincaré inequality of the type

$$\int_\Omega |u|^p\, w\, \mathrm{d}x \le C \int_\Omega |\mathrm{D}u|^p\, w\, \mathrm{d}x\,, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, whenever w is yet again an A_p weight; see Chiarenza–Frasca's streamlined proof in [11]. By today, the class of admissible weights is known to be even larger [38].

Muckenhoupt weights have also made an impact on the field of potential theory. The classical Dirichlet problem on a domain $\Omega \subset \mathbb{R}^n$ is to look for a function u such that

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega\,, \\ u|_{\partial\Omega} = f, \end{cases} \tag{1.6}$$

where the boundary data f belongs to a space such as $C(\partial\Omega)$ or $L^p(\partial\Omega)$. Roughly speaking, the regularity of the domain Ω determines the smoothness of the solution, and harmonic measure carries information on that regularity. It follows from the work of Dahlberg [19, 20] that the Dirichlet problem (1.6) with boundary data in L^p is uniquely solvable with the solution having nontangential boundary values in L^p, whenever the harmonic measure is an A_∞ weight on the domain boundary, and the Poisson kernel associated with the measure satisfies a reverse Hölder inequality. On harmonic measure see also [8, 45].

We will be assuming a degree of familiarity with the Euclidean theory of weights. Important general references on the topic, mostly on Euclidean spaces, are [23, 29, 34, 35, 61, 63, 64]. Weights on metric measure spaces are further discussed in Chapter 3.1. The lecture note [24] takes a modern approach to Muckenhoupt weights and might be recommended as an introduction.

1.2 Notation

We begin by introducing notation and definitions that are either used throughout or would clutter the presentation elsewhere. Whenever $E \subset X$ is a measurable subset and the function f is Lipschitz continuous on every compact subset of E, we say that f is locally Lipschitz on E, denoted $f \in \mathrm{Lip}_{\mathrm{loc}}(E)$. Locally integrable functions $L^1_{\mathrm{loc}}(E)$ are defined analogously.

If the measure ν is absolutely continuous with respect to μ and if there exists a nonnegative locally integrable function w such that $d\nu = w\, \mathrm{d}\mu$, we call ν a weighted measure with respect to μ, and w a weight, following [63, p. 1]. Habitually, we abuse notation and do not distinguish between the measure and the weight function. Consequently, for any measurable subset $F \subset E$ and weight w on E, we write $w(F) = \int_F w\, \mathrm{d}\mu$.

The integral average of a function $f \in L^1(E)$ over a measurable set $F \subset E$, with $0 < \mu(F) < \infty$, is abbreviated

$$\frac{1}{\mu(F)} \int_F f \,\mathrm{d}\mu = \fint_F f \,\mathrm{d}\mu = f_F.$$

Averages with respect to a weighted measure are occasionally abbreviated

$$\frac{1}{w(F)} \int_F fw \,\mathrm{d}\mu = f_{w,F}.$$

An open ball with center $x \in X$ and radius $r > 0$ is denoted $B = B(x, r)$, where the center and radius are left out when not relevant to the discussion. We write $\mathrm{r}(B) = r$, and $aB = B(x, ar)$ for the ball dilated by a constant $a > 0$.

When $X = \mathbb{R}^n$, the role of balls is largely taken over by cubes. A cube $Q \subset \mathbb{R}^n$ is a half-open set of the form

$$Q = [a_1, b_1) \times \ldots \times [a_n, b_n)$$

with $b_1 - a_1 = \ldots = b_n - a_n$. Imitating the notation for balls, a cube $Q = Q(x, r)$ is determined by its midpoint x and side length $\mathrm{l}(Q) = 2r = b_1 - a_1$. We also adopt the shorthand notation $aQ = Q(x, ar)$ for dilated cubes.

In Chapter 2.2 we will encounter a wealth of vocabulary pertaining to dyadic cubes. These are constructed as subcubes of a fixed cube $Q_0 \subset \mathbb{R}^n$. The collection of its dyadic children, denoted $\mathrm{ch}_{\mathcal{D}}(Q_0)$, are the 2^n cubes with side length $\mathrm{l}(Q_0)/2$ obtained by bisecting each edge. Continuing this process recursively, we obtain the infinite collection $\mathcal{D}(Q_0)$ of dyadic subcubes, that consists of Q_0 and its dyadic descendants in every generation. Each cube Q in $\mathcal{D}(Q_0) \setminus \{Q_0\}$ has a unique dyadic parent, denoted πQ: the cube $Q' \in \mathcal{D}(Q_0)$ such that $Q \in \mathrm{ch}_{\mathcal{D}}(Q')$.

With respect to a generic collection $\mathcal{E} \subset \mathcal{D}(Q_0)$ of cubes with $Q_0 \in \mathcal{E}$, the $\mathcal{E}$-parent $\pi_{\mathcal{E}} Q$ of a cube $Q \subset Q_0$ is the minimal cube in $\mathcal{E}$ that contains Q. The inclusion need not be strict, so $\pi_{\mathcal{E}} Q = Q$ whenever $Q \in \mathcal{E}$. The $\mathcal{E}$-children $\mathrm{ch}_{\mathcal{E}}(Q)$ of a cube $Q \in \mathcal{E}$ are the maximal cubes in $\mathcal{E}$ strictly contained in Q.

The classical Hardy–Littlewood maximal function is given by

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \,\mathrm{d}\mu, \tag{1.7}$$

where X is a metric space, B balls in X, and $f \in L^1_{\mathrm{loc}}(X)$. When $X = \mathbb{R}^n$, cubes Q replace balls in the definition.

Various constants are denoted by the letter C, whose dependence on parameters may be indicated in parentheses. Many careful descriptions of constants have been omitted from this presentation, and the interested reader will do well to consult the original publication.

2. Weighted norm inequalities

2.1 Approaches to locality: Intrinsic methods

The attentive reader of Chapter 1.1 may have noticed an inconsistency in the presentation. Namely, in the context of harmonic analysis, one is typically concerned with global properties, while partial differential equations tend to be solved within bounded domains. Compare, for instance, the weighted inequalities (1.4) and (1.5). This observation does reflect the situation at large. While the global theory of A_p weights is well established, especially in $\mathbb{R}^n$, interest in local weights has been limited to solitary articles such as [3, 40].

Our contribution to the study of local weights is Publication I (PI), whose principal results have since been included in the book [46]. The main result is a two-weight Sobolev–Poincaré inequality for weights defined on a bounded domain Ω. Precisely, we show that for any locally Lipschitz continuous function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$

$$\left(\inf_{c\in\mathbb{R}^n} \int_\Omega |u-c|^q \, w \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_\Omega |\nabla u|^p \, v \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2.1}$$

on condition that $\Omega \subset \mathbb{R}^n$ is a Boman domain, $1 < p \leq q < \infty$, and the weights w and $\sigma = v^{1/p-1}$ each satisfy a dyadic A_∞ condition on dyadic subcubes of the dilated cube $Q^* = \frac{9}{8}Q$, where Q is a cube in the Whitney decomposition $\mathcal{W}(\Omega)$ of Ω. Precisely, there exist constants $C_w, \delta_w > 0$ such that for all Q^*-dyadic cubes $R \subset Q^*$ and measurable sets $E \subset R$

$$\frac{w(E)}{w(R)} \leq C_w\left(\frac{|E|}{|R|}\right)^{\delta_w}, \tag{2.2}$$

and similarly for σ; we denote $w, \sigma \in A^d_\infty(Q^*)$. In addition, w and σ must comply to the following dyadic Muckenhoupt-type compatibility condition:

there exists a constant $K > 0$ such that

$$\left(\frac{1}{|R|^{1-1/n}}\right)^{p} w(R)^{\frac{p}{q}}\sigma(R)^{p-1} \leq K. \tag{2.3}$$

The standard construction of Whitney cubes can be found e.g. in [35, Appendix J]. The significance of Boman domains will be discussed shortly. For the statement of the theorem in full detail see PI, Theorem 7.1. Comparable two-weight inequalities and compatibility conditions have been established by Chanillo et al [9, 10] and Chua [13].

The proof consists of two stages. To begin with, we show the local inequality

$$\left(\int_{Q_0} \left|u - u_{Q_0}\right|^q w \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{Q_0} |\nabla u|^p v \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2.4}$$

where Q_0 is a fixed cube in $\mathbb{R}^n$, on condition that $w, \sigma \in A_\infty^d(Q_0)$, and (2.3) is satisfied in dyadic subcubes $R \subset Q_0$. This is done by the *sparse domination* method, which will be presented separately in Section 2.2. The key feature of the local inequality is that it is strictly local: all assumptions on the weights (v, w) are made within the fixed cube Q_0. This is why, in the next stage, we are able to build an estimate from the inside out, without reference to data from outside the domain.

From here, we propagate the local inequality (2.4) to the entire domain Ω, producing (2.1). This local-to-global step is where the geometry of Boman domains comes into play. In a Boman domain $\Omega \subset \mathbb{R}^n$, every two Whitney cubes Q_0 and Q_k can be joined by a chain $\mathcal{C}(Q) = (Q_0, Q_1, \ldots, Q_k)$ of Whitney cubes such that for each j there exists a cube $R \subset Q_j^* \cap Q_{j-1}^*$ for which

$$\mathrm{l}(R) \geq C(n) \max\left\{l(Q_j^*), l(Q_{j-1}^*)\right\}.$$

The formal definition of a Boman domain can be found in PI, Section 6. This class of domains was introduced in [5]. Open cubes, balls, and bounded Lipschitz domains are Boman domains in $\mathbb{R}^n$. More generally, a Euclidean domain Ω is a Boman domain if and only if it is a John domain. Unlike Lipschitz domains, John domains are allowed to have twisting cones. The relevance of these classes is covered in [6].

The main result connecting cubewise and global estimates is the following theorem. Its proof is based on an idea of Iwaniec–Nolder's [42, Lemma 4]. While technical, the proof boils down to applying elementary properties of chains. The cube $Q_0 \in \mathcal{W}(\Omega)$ is chosen to be the fixed central cube in the chain decomposition of Ω, that is, the collection $\{\mathcal{C}(Q) : Q \in \mathcal{W}(\Omega)\}$ of chains from Q_0 to Q.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a Boman domain, and w doubling weight in Ω. If $u \in L^1_{\mathrm{loc}}(\Omega, w)$ and $1 \leq p < \infty$, then*

$$\int_\Omega \left|u - u_{w,Q_0^*}\right|^p w \,\mathrm{d}x \leq C \sum_{Q \in \mathcal{W}(\Omega)} \int_{Q^*} \left|u - u_{w,Q^*}\right|^p w \,\mathrm{d}x. \tag{2.5}$$

The final steps in the proof of (2.1) are applying (2.5), Hölder's inequality, (2.4), the fact that $q \geq p$, and that the dilated Whitney cubes Q^* have bounded overlap: $\sum_{Q \in \mathcal{W}} \chi_{Q^*} \leq C(n)$.

$$
\begin{aligned}
\inf_{c \in \mathbb{R}^n} \int_\Omega |u - c|^q \, w \, \mathrm{d}x &\leq C \sum_{Q \in \mathcal{W}(\Omega)} \int_{Q^*} \left| u - u_{w,Q^*} \right|^q w \, \mathrm{d}x \\
&\leq C \sum_{Q \in \mathcal{W}(\Omega)} 2^q \int_{Q^*} \left| u - u_{Q^*} \right|^q w \, \mathrm{d}x \leq C \sum_{Q \in \mathcal{W}(\Omega)} \left(\int_{Q^*} |\nabla u|^p \, v \, \mathrm{d}x \right)^{\frac{q}{p}} \\
&\leq C \left(\sum_{Q \in \mathcal{W}(\Omega)} \int_{Q^*} |\nabla u|^p \, v \, \mathrm{d}x \right)^{\frac{q}{p}} \leq C \left(\int_\Omega |\nabla u|^p \, v \, \mathrm{d}x \right)^{\frac{q}{p}} .
\end{aligned}
$$

As an application of (2.1) we show two Sobolev-Poincaré inequalities where $v = 1$ and w is a distance weight, respectively

$$
\begin{aligned}
&d(x, E)^{-n + \frac{q}{p}(n-p)}, \text{ where } E \subset \mathbb{R}^n \text{ is a nonempty closed set, and} \\
&d(x, \partial\Omega)^{-n + \frac{q}{p}(n-p)}, \text{ where } \Omega \text{ is a Boman domain.} \qquad (2.6)
\end{aligned}
$$

In both cases $1 < p \leq q \leq \frac{np}{n-p} < \infty$, the Assouad dimension of the set E is bounded by $\frac{q}{p}(n-p)$, and the weight (2.6) satisfies a doubling condition of the type

$$
w\,(\Omega \cap Q(x, 2r)) \leq C_d w\,(\Omega \cap Q(x, r))
$$

where $Q(x, r) \subset \mathbb{R}^n$ is a cube with its midpoint $x \in \Omega$. The full statements are Theorems 7.2–7.3 in PI, which have been adapted into Theorems 10.29–10.30 in [46].

2.2 Outline of the sparse domination argument

The idea of sparse domination is to use "sparse" dyadic operators to control a continuous operator pointwise. Consequently, the original problem is reduced to showing a uniform weighted norm inequality for a significantly simpler class of dyadic operators. Dyadic techniques have been influential in contemporary harmonic analysis, Hytönen's resolution of the A_2 conjecture [41] being perhaps the most famous application. We refer to Pereyra's lecture notes [55], modestly titled *Sparse revolution,* for a survey.

In PI, we apply the sparse domination argument twice in the proof of the local inequality (2.4). This proof will serve here to demonstrate the principles underlying the sparse domination paradigm. We operate within a fixed cube $Q_0 \subset \mathbb{R}^n$ and its dyadic decomposition $\mathcal{D}(Q_0)$; recall the construction in Chapter 1.2. In the first stage, we follow the idea in [48] and show for every Lebesgue point $x \in Q_0$ of f the pointwise estimate

$$\left|f(x) - f_{Q_0}\right| \leq C \sum_{Q \in \mathcal{S}} \chi_Q(x) \fint_Q \left|f - f_Q\right| \, \mathrm{d}y,$$

where the collection $\mathcal{S}$ of dyadic cubes inside Q_0 depends on f. The letter S stands for *sparse* collection in the sense that there are pairwise disjoint, measurable subsets $E_S \subset S \in \mathcal{S}$, each of which has a large w-measure compared to that of S. Lemma 2.2.1 describes the candidate for $\mathcal{S}$, whose sparsity is subsequently quantified by Lemma 2.2.2.

Lemma 2.2.1. *Let Q_0 be a cube in $\mathbb{R}^n$, $f \in L^1(Q_0)$, and $w \in A_\infty^d(Q_0)$ with $C_w, \delta_w > 0$. There exists a collection $\mathcal{S}$ of dyadic cubes and a constant $\rho = \rho(C_w, \delta_w) > 1$ such that for each cube $S \in \mathcal{S}$*

(a) *if $Q \subset Q_0$ is dyadic cube such that $\pi_{\mathcal{S}} Q = S$, then*

$$\fint_Q |f - f_S| \, \mathrm{d}x \leq \rho \fint_S |f - f_S| \, \mathrm{d}x;$$

(b)
$$\sum_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} w(S') \leq C_w \rho^{-\delta_w} w(S) < w(S).$$

The collection $\mathcal{S}$ is constructed by a stopping-time argument. Fix a function $f \in L^1(Q_0)$ and a constant ρ such that $C_w \rho^{-\delta_w} < 1$. We place Q_0 inside $\mathcal{S}$ and proceed recursively: for each dyadic cube $S \in \mathcal{S}$, we add to $\mathcal{S}$ the maximal dyadic cubes $S' \subset S$ that satisfy the stopping condition

$$\fint_{S'} |f - f_S| \, \mathrm{d}x > \rho \fint_S |f - f_S| \, \mathrm{d}x.$$

This process is iterated *ad infinitum* if necessary. As a result, we obtain a collection $\mathcal{S}$ of dyadic cubes in Q_0 that satisfies (a). The inequality (b) follows from the stopping rule and the $A_\infty^d(Q_0)$ condition for w.

Next, we build another collection of sets $E_S \subset S$, pairwise disjoint yet occupying a large part of the weighted measure of each $S \in \mathcal{S}$.

Lemma 2.2.2. *Let Q_0 be a cube in $\mathbb{R}^n$, $w \in A_\infty^d(Q_0)$ with constants $C_w, \delta_w > 0$, and $f \in L^1(Q_0)$. There exists a collection $\mathcal{S}$ of dyadic cubes in Q_0 satisfying the following conditions.*

(a) *There is a constant $\eta = \eta(C_w, \delta_w) > 0$ and a collection $\{E_S : S \in \mathcal{S}\}$ of pairwise disjoint sets such that for every $Q \in \mathcal{S}$, E_S is a measurable subset of S with $w(E_S) \geq \eta w(S)$;*

(b) *For every Lebesgue point $x \in Q_0$ of f,*

$$\left|f(x) - f_{Q_0}\right| \leq C \sum_{S \in \mathcal{S}} \chi_S(x) \fint_S |f - f_S| \, \mathrm{d}y. \tag{2.7}$$

Beginning with the collection $\mathcal{S} \subset \mathcal{D}(Q_0)$ from Lemma 2.2.1, we are going to construct the collection $\{E_S : S \in \mathcal{S}\}$ by removing selected parts of the cubes $S \in \mathcal{S}$, namely their $\mathcal{S}$-children. For every $S \in \mathcal{S}$, let

$$E_S = S \setminus \bigcup_{S' \in \operatorname{ch}_{\mathcal{S}}(S)} S'. \tag{2.8}$$

The task is now to show that $\{E_S\}$ is the collection postulated by Lemma 2.2.2. Disjointness of the sets E_S follows from the dyadic decomposition. Furthermore, for a fixed $S \in \mathcal{S}$, (2.7) implies that

$$w(E_S) = w(S) - \sum_{S' \in \operatorname{ch}_{\mathcal{S}}(S)} w(S') \geq \left(1 - C_w \rho^{-\delta_w}\right) w(S),$$

whereby (a) is verified. The content of claim (b) is that the quantity $\left|f - f_{Q_0}\right|$ is dominated pointwise by the *dyadic sparse operator* on the right-hand side of (2.7). The idea of the proof is to express the left-hand side in terms of dyadic differences of the type

$$\left|f' - f_{Q_0}\right| \chi_{Q_0} \leq \sum_{S \in \mathcal{S}} \left| \sum_{Q : \pi_{\mathcal{S}} Q = S} \sum_{Q' \in \operatorname{ch}_{\mathcal{D}}(Q)} \chi_{Q'} \left(f_{Q'} - f_Q\right) \right|.$$

The double sum on each S is then split among E_S and $S \setminus E_S$, and is found to collapse on both sets by virtue of the nested dyadic structure, whereby we find (2.7).

We aim to control the right-hand side of (2.7) by duality and maximal function arguments. Lemma 2.2.2 is used to show a localized and weighted variant of the Fefferman–Stein inequality (see [27] and [63, Theorem III.3]), that is of independent interest. For a cube Q_0 and $f \in L^1(Q_0)$, the *dyadic sharp maximal function* is given by

$$M^{d,\sharp}_{Q_0} f(x) = \sup_{\substack{Q \subset Q_0 \\ x \ni Q}} \fint_Q \left|f - f_Q\right| \mathrm{d}y,$$

where the supremum is taken over all dyadic cubes $Q \subset Q_0$ such that $x \in Q$.

Theorem 2.2.3. *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $1 < p < \infty$, $w \in A^d_\infty(Q_0)$, and $f \in L^1(Q_0)$. Then*

$$\int_{Q_0} \left|f - f_{Q_0}\right|^p w \,\mathrm{d}x \leq C \int_{Q_0} \left(M^{d,\sharp}_{Q_0} f\right)^p w \,\mathrm{d}x.$$

To see this, rewrite the left-hand side using Lemma 2.2.2 (b). The pth root of the ensuing integral can then be estimated by duality.

A Poincaré inequality on cubes [33, p. 164] shows that for a function $u \in \operatorname{Lip}(Q_0)$, the dyadic sharp maximal function is controlled by the *dyadic* *fractional maximal function* of the gradient. For $0 \le \alpha < n$, this function is given by

$$M^d_{\alpha,Q_0} f(x) = \sup_{\substack{Q\subset Q_0\\ Q\ni x}} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f| \,\mathrm{d}y,$$

where the supremum is again taken over all dyadic cubes $Q \subset Q_0$ such that $x \in Q$.

To estimate the dyadic fractional maximal function we will be needing another sparse domination argument, that is Lemma 5.1 in PI. This step is also where we change weights, and the compatibility condition (2.3) appears. The sparse estimate for almost every $x \in Q_0$ takes the form

$$\left(M^d_{\alpha,Q_0} f(x)\right)^p \le C \sum_{S\in\mathcal{S}} \chi_S(x) \left(\frac{1}{|S|^{1-\alpha/n}} \int_S |f| \,\mathrm{d}y \right)^p, \tag{2.9}$$

and the stopping rule generating the collection $\mathcal{S}$ is as follows. Let k_0 be the smallest integer satisfying

$$\frac{1}{|Q_0|^{1-\alpha/n}} \int_{Q_0} |f| \,\mathrm{d}x \le a^{k_0},$$

where $a > 2^n$ depends on the dyadic $A^d_\infty(Q_0)$ constants of the weight σ, and no longer on w. Let $\mathcal{S}_{k_0} = \{Q_0\}$, and $\mathcal{S}_k$ for $k > k_0$ be the collection of maximal dyadic cubes $Q \subset Q_0$ satisfying

$$a^k < \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f| \,\mathrm{d}x.$$

Define sets $E_{k,Q}$ by removing from Q the $(k+1)$th step of the stopping construction:

$$E_{k,Q} = Q \setminus \bigcup_{R\in\mathcal{S}_{k+1}} R.$$

The collection $\left\{E_{k,Q} : k \ge k_0\,,\, Q \in \mathcal{S}_k\right\}$ is the "heavy" collection that we seek. By means of the nested dyadic structure and the stopping condition, we have in fact pinned down the level sets $\{x \in Q_0 : a^k < M^d_{\alpha,Q_0} f(x) \le a^{k+1}\}$ of the maximal function. This is by no means unexpected, because the maximal function was dyadic to begin with.

The sparse estimate (2.9) leads to a two-weight inequality for the fractional maximal function, which is a localized variant of [56, Theorem 1.1].

Theorem 2.2.4. *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $0 \le \alpha < n$, $1 < p \le q < \infty$, and (v, w) a pair of weights in Q_0 such that $\sigma = v^{-1/(p-1)} \in A^d_\infty(Q_0)$. The following conditions are equivalent.*

(a) *There exists a $C > 0$ such that, for all $f \in L^1(Q_0)$,*

$$\left(\int_{Q_0} \left(M^d_{\alpha,Q_0} f\right)^q w \,\mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\int_{Q_0} |f|^p v \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

(b) *There exists a $K > 0$ such that, for all dyadic cubes $Q \subset Q_0$,*

$$\left(\frac{1}{|Q|^{1-\alpha/n}}\right)^p w(Q)^{\frac{p}{q}} \sigma(Q)^{p-1} \leq K.$$

With these main results in place, we are finally in a position to prove the local inequality (2.4). We first apply Theorem 2.2.3, then a Poincaré inequality for the dyadic fractional maximal function (PI, Lemma 4.4) and finally Theorem 2.2.4:

$$\left(\int_{Q_0} |u - u_{Q_0}|^q \, w \, \mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\int_{Q_0} \left(M_{Q_0}^{d,\sharp} u\right)^q w \, \mathrm{d}x\right)^{\frac{1}{q}}$$

$$\leq C \left(\int_{Q_0} \left(M_{1,Q_0}^{d} |\nabla u|\right)^q w \, \mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\int_{Q_0} |\nabla u|^p \, v \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

The local inequality has an independent application to weak supersolutions of the p-Laplace equation

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0 \quad \text{in } \Omega.$$

Recall that $W^{1,p}_{\mathrm{loc}}$ is the Sobolev space of all functions in L^p_{loc} whose distributional first derivatives lie in L^p_{loc}. We call $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a weak supersolution in Ω if for all nonnegative $\eta \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \eta(x) \, \mathrm{d}x \geq 0.$$

As per [38, Theorem 3.59], nonnegative weak supersolutions are local A_1 weights in cubes "well inside" Ω. This fact enables us to apply (2.4) to obtain the following single-weighted Poincaré inequality, Theorem 7.7 in PI.

Theorem 2.2.5. *Let Ω be a bounded domain, and $\frac{2n}{n+1} < p < \infty$. Let $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak supersolution of the p-Laplace equation in Ω such that $w(x) > 0$ for almost every $x \in \Omega$, and $Q_0 \subset \Omega$ a cube such that $4Q_0 \subset \Omega$. The weighted Poincaré inequality*

$$\int_{Q_0} |u - u_{Q_0}|^p \, w \, \mathrm{d}x \leq C \, \mathrm{l}(Q_0)^p \int_{Q_0} |\nabla u|^p \, w \, \mathrm{d}x$$

holds for every $u \in \mathrm{Lip}(Q_0)$ with $C = C(n,p) > 0$.

Letting $v = |Q_0|^{p/n} \, w$, $\sigma = v^{-1/(p-1)}$, $q = p$, and Q_0 such that $4Q_0 \subset \Omega$, the proof consists in verifying that $w, \sigma \in A^d_\infty(Q_0)$ and that (v, σ) satisfy the compatibility condition of Theorem 2.2.4 (b).

3. Muckenhoupt weights on metric spaces

3.1 Setting up the space

A typical metric environment to study Muckenhoupt weights is called a *space of homogeneous type* in the sense of [15, Chapitre III]. This is essentially a space (X, d, μ) with a quasimetric (satisfying a relaxed triangle inequality) and a doubling property: there exists a constant $C > 1$ only depending on μ such that for all balls $B \subset X$

$$0 < \mu(2B) \leq C\mu(B) < \infty. \tag{3.1}$$

These assumptions provide us with enough structure to reproduce the Euclidean theory in its essentials. Strömberg–Torchinsky [63] investigate and, indeed, demonstrate the importance of the doubling condition. It turns out that once the underlying measure is doubling, the A_p condition (1.1) for some $p > 1$ implies several relevant properties of A_p weights, such as the reverse Hölder inequality and comparability of $\mathrm{d}\nu = w\,\mathrm{d}\mu$ to the underlying measure [63, Theorem 15]. A Borel measure ν is said to be comparable to the measure μ if there exist constants $0 < \eta, \varepsilon < 1$ such that for any ball $B \subset X$ and measurable subset $E \subset B$,

$$\mu(E) \leq \varepsilon\mu(B) \quad \text{implies that} \quad \nu(E) \leq \eta\nu(B). \tag{3.2}$$

We will be working in a metric measure space (X, d, μ), where the nontrivial Borel measure μ satisfies the doubling property (3.1). The doubling condition implies that the space is separable, and we further assume it to be complete and hence proper [4, Proposition 3.1]. Unlike in Euclidean spaces, we will not be able to benefit from a convex, nested, dyadic structure. The most obvious candidates to replace dyadic cubes are Vitali- and Whitney-type coverings with balls, and generalized dyadic sets introduced by Christ [12]. Dyadic sets share many good properties with their Euclidean counterparts, such as having exactly one parent and a determined number of children, but we have very little to say about their “shape”, including connectedness and convexity. Whitney balls, on the other hand, are not pairwise disjoint, although they have bounded overlap. However, as long as the space supports a doubling measure we can find a Calderón-Zygmund decomposition, for instance. We adopt the Whitney decomposition from [36, Lemma 2.8]; see Publication II, Lemma 3.3.

As for the class $A_\infty(X)$, we will be referring to the following characterization, sometimes called the reverse Jensen inequality:

$$[w]_\infty = \sup_{\substack{B\subset X\\ B\ni x}} \left(\fint_B w \,\mathrm{d}\mu \right) \exp\left(-\fint_B \log w \,\mathrm{d}\mu \right) < \infty, \tag{3.3}$$

where $w \in L^1_{\mathrm{loc}}(X)$ is a nonnegative function. In Euclidean spaces it is well known that this inequality and (2.2) describe the same class of weights, albeit dyadic in the case of the latter; see, for instance, [29, Theorem IV.2.15]. The situation in more general metric measure spaces will be discussed in Chapter 3.3.

Considering harmonic analysis on metric measure spaces, Coifman and Weiss' book [15] turns out to have been ahead of its time. Fifty years later, a good modern textbook remains to be written. The principal reference to weight theory on metric measure spaces is the aforementioned [63]. Technical tools can be found in [31, 36]. While not directly relevant to us, [22] is a book on Littlewood–Paley theory and wavelets on spaces of homogeneous type, that draws on [15] for much of its basics.

3.2 Approaches to locality: Extension

Returning to the topic of local weights, an alternative to intrinsic methods discussed in Chapter 2 is to borrow global results by finding an extension that coincides with the original weight on the set $E \subset X$ in question. In particular, we would like to investigate those subsets E for which an extension can be found. The problem setting is similar to that of Sobolev extensions, on which there exists an extensive body of research; see e.g. [37, 44].

Publication II (PII) takes a step in this direction. Namely, we prove the following theorem, that is the generalization of a Euclidean result attributed to Thomas H. Wolff.

Theorem 3.2.1. *Let X be a complete metric space with a doubling measure, $E \subset X$ a measurable set with $\mu(E) > 0$, and w a weight on E. Then, for $1 < p < \infty$, the following statements are equivalent.*

(i) *There exists a weight $W \in A_p(X)$ such that $W = w$ a. e. on E;*

(ii) *There exists an* $\varepsilon > 0$ *such that*

$$\sup_{B\subset X}\left(\frac{1}{\mu(B)}\int_{B\cap E} w^{1+\varepsilon}\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(B)}\int_{B\cap E}\left(\frac{1}{w^{1+\varepsilon}}\right)^{\frac{1}{p-1}}\mathrm{d}\mu\right)^{p-1} < \infty.$$

In addition, whenever $p = 1$*, the condition* (ii) *takes the following form: There exists a constant* $C > 0$ *such that*

$$\frac{1}{\mu(B)}\int_{B\cap E} w^{1+\varepsilon}\,\mathrm{d}\mu \le C \operatorname*{ess\,inf}_{B\cap E} w^{1+\varepsilon}$$

for every ball $B \subset X$.

Comparing (ii) to the classical A_p condition, it is clear that we need to deal with weights and maximal functions restricted to arbitrary measurable subsets $E \subset X$. We have chosen to call these classes *induced* A_p *weights* on E, denoted $\widetilde{A}_p(E)$.

Definition 3.2.2. On a metric space X, let $E \subset X$ be a measurable subset with $\mu(E) > 0$. Let w be a weight on E. If $1 < p < \infty$, we say that $w \in \widetilde{A}_p(E)$ whenever

$$[\![w]\!]_p = \sup_{B\subset X}\left(\frac{1}{\mu(B)}\int_{B\cap E} w\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(B)}\int_{B\cap E}\left(\frac{1}{w}\right)^{\frac{1}{p-1}}\mathrm{d}\mu\right)^{p-1} < \infty.$$

If $p = 1$, we define $\widetilde{A}_1(E)$ as the class of weights w for which there exists $C > 0$ with

$$\frac{1}{\mu(B)}\int_{B\cap E} w\,\mathrm{d}\mu \le C \operatorname*{ess\,inf}_{B\cap E} w$$

for every ball $B \subset X$. We denote by $[\![w]\!]_1$ the infimum of the $C > 0$ for which this inequality holds.

In other words, the extension Theorem 3.2.1 states that a weight w, initially defined on a subset $E \subset X$, possesses an extension to the whole space whenever $w^{1+\varepsilon}$ is an induced weight on E.

We need to assume the $\widetilde{A}_p$ condition for $w^{1+\epsilon}$ instead of simply stating the corresponding condition for w, because it is unclear whether the induced weights satisfy a self-improving property. This results from the elementary fact that when E is an arbitrary subset, we are unable to control the measures of the *relative balls* $B \cap E$. Even when the measure is positive, $\mu(B \cap E)$ might be very small in comparison to $\mu(B)$, not to mention the doubling property: $\mu(2B \cap E)$ or $w(2B \cap E)$ are not comparable to $\mu(B \cap E)$ or $w(B \cap E)$ in general. As a consequence, we cannot ensure that w and $w^{-1/(p-1)}$ satisfy a reverse Hölder inequality, and hence the Gehring lemma cannot be used to obtain the self-improving property. See [63] and the discussion in Chapter 3.3.

The proof of Theorem 3.2.1 rests on three classical results: Jones factorization, Coifman–Rochberg lemma, and the self-improving property of $A_p(X)$ weights. Owing to the dearth of references on A_p weights in metric measure spaces, we have chosen to prove each theorem in detail in PII. Our version of the Jones factorization theorem involves a power of an $\widetilde{A}_p(E)$ weight.

Theorem 3.2.3. *Let $E \subset X$ be a measurable set with $\mu(E) > 0$, $p > 1$, and v a weight on E such that $v^r \in \widetilde{A}_p(E)$ for some $r > 1$. Then there exist weights $v_1, v_2 \in \widetilde{A}_1(E)$ such that $v = v_1 v_2^{1-p}$.*

Using elementary properties of $\widetilde{A}_p(E)$ weights, we show that whenever $v^r \in \widetilde{A}_p(E)$ for some $r > 1$, then v and $v^{-1/(p-1)}$ are induced weights on E of classes $q_1 < p$ and $q_2 < p'$, respectively. Here p' denotes the conjugate exponent of p such that $1/p + 1/p' = 1$. It follows that the maximal function relative to the set E, given by

$$m_E f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_{B \cap E} |f| \, \mathrm{d}\mu,$$

is bounded both on $L^p(E, v)$ and $L^{p'}\left(E, v^{-1/(p-1)}\right)$, and we can follow the track of the classical proof [34, Theorem 7.5.1] (in $\mathbb{R}^n$, but the technique is the same).

Ultimately, we would like to characterize those subsets $E \subset X$ from which extension is possible, and preferably in geometric terms. Theorem 3.2.1 prompts us to translate the $\widetilde{A}_p(E)$ condition into a geometric condition on the set E. In this sense, the extension problem would appear to share certain similarities with local-to-global problems such as the one in Chapter 2.1.

Holden [39] has verified Theorem 3.2.1 (ii) in $\mathbb{R}^n$ under additional assumptions on the set E that arise from his argument, yet we were unable to reproduce his proof even in $\mathbb{R}^n$. The reason is yet again the difficulty of controlling the measures of the sets $B \cap E$ (or $Q \cap E$ when $X = \mathbb{R}^n$). In the end, it is not clear what geometric assumptions to make on the set E so as to overcome this problem and successfully verify the $\widetilde{A}_p(E)$ condition. Publication II provides a starting point for further research.

3.3 From Muckenhoupt to reverse Hölder

A well-known and remarkable property of A_p weights is that they satisfy a reverse Hölder inequality (RHI): whenever $\Omega \subset X$ is an open subset, there exist $p > 1$ and a constant C such that for every ball $B \Subset \Omega$

$$\left(\fint_B w^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \le C \fint_B w \, \mathrm{d}\mu. \tag{3.4}$$

Here and below, the relation $B \Subset \Omega$ indicates that the closure of B is a compact subset of Ω. If a function $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies the reverse Hölder inequality with exponent $p > 1$, we say that f belongs to the *p-reverse Hölder class,* denoted $f \in RH_p(\Omega)$. These functions were first studied by Gehring [30] and Coifman–Fefferman [14].

In Euclidean spaces, the reverse Hölder inequality characterizes A_p weights, in the sense that a weight w belongs to A_p for some p if and only if it satisfies a reverse Hölder inequality for some $s > 1$. In metric measure spaces, even those with a doubling measure, this is generally not the case. Namely, a reverse Hölder inequality need not imply the A_p condition (1.1), as shown by [63, Theorem 15]. If we wish to recover the RHI as a characterization of A_p weights, we need to make another assumption on the setting. Strömberg and Torchinsky [63] develop the A_p–reverse Hölder theory in metric measure spaces under the assumption that the measure of a ball depend continuously on its radius, while Kinnunen and Shukla [47, 59] assume the *α-annular decay property.*

Definition 3.3.1. A metric measure space (X, d, μ) with a doubling measure μ is said to satisfy the α-annular decay property with $0 \leq \alpha \leq 1$, if there exists $C \geq 1$ such that for every $x \in X$, $r > 0$, and $0 < \delta < 1$

$$\mu\left(B(x, r) \setminus B(x, (1-\delta)r)\right) \leq C\delta^{\alpha} \mu(B(x, r)). \tag{3.5}$$

The constant C is independent of the point, radius, and δ.

Whenever the exact parameter α is not of interest, we say that a space satisfies an annular decay property if it satisfies the α-annular decay property for some $0 \leq \alpha \leq 1$.

In [47, 59] the annular decay property guarantees that when the underlying measure is doubling, any weighted measure comparable to it is doubling as well. It follows that the comparability of measures (3.2), reverse Jensen (3.3), and reverse Hölder (3.4) inequalities can be taken as equivalent characterizations of the class A_∞, just as in the Euclidean case. In short, it begins to make sense to state that

$$A_\infty = \bigcup_{p \geq 1} A_p.$$

At present, it is not clear what is the minimal necessary assumption to reproduce the A_p–reverse Hölder connection on metric spaces. The annular decay property is certainly sufficient, and enjoys the advantages of being well known, fully quantitative, as well as formulated in terms of the measure μ alone. For convenience, we will adopt the assumption (3.5) wherever a strong reverse Hölder inequality is needed. Early publications involving the annular decay property or a slight variant are [7, 16, 21]. It is well known to hold true for a fairly large class of spaces, including all length spaces; see e.g. [7, 57].

The annular decay property unlocks the very important Gehring lemma, which states that a uniform reverse Hölder inequality is self-improving: for a function $f \in RH_p$, there exists $q > p$ such that $f \in RH_q$. Moreover, without an additional assumption such as the annular decay property, one only obtains a *weak* RHI with exponent q, that will be introduced shortly. For two different proofs of this lemma in metric measure spaces see [49, Theorem 3.1], and [59]. We will not be directly applying Gehring's lemma, but it again illustrates the perhaps unexpected phenomena that occur in metric spaces.

3.4 Weak reverse Hölder inequalities

A weak reverse Hölder inequality involves an increasing support on the right-hand side. We say that a nonnegative, locally integrable function f satisfies a weak reverse Hölder inequality on $\Omega \subset X$, whenever there exist $p > 1$ and a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$,

$$\left(\fint_B f^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \le C \fint_{2B} f \,\mathrm{d}\mu.$$

Such functions constitute the weak p-reverse Hölder class $WRH_p(\Omega)$. This class is a genuine relaxation of RH_p (3.4) in that functions in WRH_p are no longer necessarily doubling. Furthermore, they may reach zero on a set of nonzero measure, unlike Muckenhoupt weights. Weak reverse Hölder inequalities arise naturally in Caccioppoli estimates for quasiregular mappings and nonlinear partial differential equations, such as in the early articles [50, 32], while in [62] it was established that they also self-improve. See also [4, Chapter 3.5] on Gehring's lemma, in fact the weak version.

Much like in the strong case, the class WRH_p can be characterized in several different ways. In Publication III (PIII), we introduce eleven characterizations in total; the following theorem highlights four of these. Statement (iv) involves functions of bounded mean oscillation (BMO). These are functions $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\|f\|_{BMO} = \sup_{B \subset \Omega} \left(\fint_B |f - f_B| \,\mathrm{d}\mu \right) < \infty.$$

Theorem 3.4.1 (selection)**.** *Let X be a metric measure space with a doubling measure, $\Omega \subset X$ an open set, and w a weight on Ω. The following statements are equivalent.*

(i) *There exist $p > 1$ and a constant $C > 0$ such that*

$$\fint_B w^p \,\mathrm{d}\mu \le C \left(\fint_{2B} w \,\mathrm{d}\mu \right)^p$$

for every ball B with $2B \Subset \Omega$;

(ii) *There exist $\eta, \varepsilon > 0$ with $\eta < C_d^{-5}$ such that if B is a ball with $2B \Subset \Omega$ and $F \subset B$ a measurable set, then $\mu(F) \leq \varepsilon\mu(B)$ implies that $w(F) \leq \eta w(2B)$;*

(iii) *There exists a constant $C > 0$ such that*

$$\int_B M(w\mathcal{X}_B)\,\mathrm{d}\mu \leq Cw(2B)$$

for every ball B with $2B \Subset \Omega$;

(iv) *There exists a constant $C > 0$ such that for every ball B with $11B \Subset \Omega$ and every function $f \in \mathrm{BMO}(\Omega)$ with $\|f\|_{\mathrm{BMO}(\Omega)} \leq 1$, it holds that*

$$\int_B |f - f_B|\, w\,\mathrm{d}\mu \leq Cw(2B).$$

As a matter of fact, the dilatation factor 2 could be replaced by any other constant $\sigma > 1$, as detailed in PIII, Theorem 4.4. Either way, no additional assumptions on the space need to be made, because the weighted measure is not required to be doubling. In particular, the annular decay assumption is not invoked. While the statements of Theorem 3.4.1 rather unsurprisingly resemble various characterizations of Muckenhoupt A_∞ weights, Publication III includes examples of other A_∞-like conditions that are not satisfied by WRH_p functions, at least not without nontrivial modifications.

The above statements (i)–(iv) in particular derive from [1, 58, 60]. Condition (ii) is, in its essence, the one we have chosen to call *qualitative nondoubling* A_∞: the nondoubling analogy of the fact that the measure induced by an A_∞ weight is comparable to the underlying measure. This is clearly the weakest of the many possible definitions of an A_∞ weight [63, Theorem 15], and remains so in the weak case. Accordingly, the most demanding part of the proof of Theorem 3.4.1 is to show that (ii) implies (i). We need the following lemma, a distributional estimate of sorts for the weight w.

Lemma 3.4.2. *Assume that w satisfies Theorem 3.4.1* (ii)*. There exist constants $\gamma > C_d^3$ and $\beta > 0$, only depending on the parameters of* (ii)*, for which the following statement holds. Let B be a ball with $2B \Subset \Omega$, $0 < r < 3/2$, and $\lambda \leq 10^{-1}$. Then*

$$\int_{rB\cap\{w\geq\gamma D\}} w\,\mathrm{d}\mu \leq \gamma^{-\beta} \int_{(r+\lambda)B\cap\{w\geq\gamma^{-1}D\}} w\,\mathrm{d}\mu,$$

where

$$D = D(\lambda, B) = \frac{w(2B)}{\mu(2B)} C_d^{\log_2 \frac{4}{5\lambda}+1}. \tag{3.6}$$

An outline of the proof follows. For an $\eta > 0$ small enough such that $\eta C_d^5 < 1$, let ε be the parameter associated with η from (ii). We choose the constants $\gamma, \beta > 0$ so that

$$\gamma > \max\left\{C_d^3, \frac{C_d^2}{\varepsilon}, \frac{1}{1-\eta C_d^5}\right\}, \quad \text{and} \quad \gamma^\beta = \frac{1-\gamma^{-1}}{\eta C_d^5}.$$

Write $I = \{y \in \Omega : w(y) \geq \gamma D\}$, and $J = \{y \in \Omega : w(y) \geq \gamma^{-1} D\}$. If r is fixed as in the assumption, and x is a Lebesgue point contained in $I \cap rB$, we denote

$$s_x = \inf\left\{s > 0 : \; B(x,s) \Subset \Omega \; \text{ and } \fint_{B(x,s)} w \, \mathrm{d}\mu \leq D\right\}, \quad \text{and} \quad r_x = \frac{s_x}{10}.$$

It follows from the choice of parameters and the doubling condition for μ that $0 < s_x < 5\lambda\, \mathrm{r}(B)$. Using the assumption (ii) in addition to the good parameters, we obtain the estimate

$$\int_{I \cap B(x, 5r_x)} w \, \mathrm{d}\mu \leq \gamma^{-\beta} \int_{J \cap B(x, r_x)} w \, \mathrm{d}\mu$$

at almost every point $x \in I \cap rB$. The Vitali covering lemma [4, Lemma 1.7] provides a disjoint family of balls $\{B(x_j, r_j)\}_j$ such that

$$\bigcup_{x \in I \cap rB} B(x, r_x) \subset \bigcup_j B(x_j, 5r_j),$$

where we have written $r_j = r_{x_j}$ for short. Note that $r_j \leq \lambda r(B)/2$, which implies that $B(x, r_x) \subset (r + \lambda)B$, and finally

$$\begin{aligned} \int_{I \cap rB} w \, \mathrm{d}\mu &\leq \sum_j \int_{I \cap B(x_j, 5r_j)} w \, \mathrm{d}\mu \leq \gamma^{-\beta} \sum_j \int_{J \cap B(x_j, r_j)} w \, \mathrm{d}\mu \\ &= \gamma^{-\beta} \int_{J \cap \bigcup_j B(x_j, r_j)} w \, \mathrm{d}\mu \leq \gamma^{-\beta} \int_{J \cap (r+\lambda)B} w \, \mathrm{d}\mu. \end{aligned}$$

Spadaro's original Euclidean proof [60, Lemma 2.1] uses the Besicovitch covering theorem, which must here be replaced by a more careful argument.

Using Lemma 3.4.2, it is now easy to complete the proof of (ii) $\Rightarrow$ (i). Let B be a ball with $2B \Subset \Omega$, γ and β as in the statement of Lemma 3.4.2, and $p > 1$ so that $2(p-1) < \beta$. Denote

$$\lambda_k = \frac{4}{5} \cdot 2^{1 - 2k \frac{\log \gamma}{\log C_d}}, \quad k = 1, 2, \ldots$$

We find that $\lambda_k < 1/10$ for every k, and that $\sum_{k=1}^\infty \lambda_k < 1/2$. Also, the corresponding constants $D_k = D(\lambda_k, B)$ given by (3.6) satisfy

$$D_k = \gamma^{2k} \frac{w(2B)}{\mu(2B)}, \quad k = 1, 2, \ldots, \quad \text{and} \quad D_k = \gamma^2 D_{k-1}, \quad k = 2, 3, \ldots$$

Then

$$\begin{aligned}\int_B w^p \,\mathrm{d}\mu &= \int_{B\cap\{w\le\gamma D_1\}} w^p \,\mathrm{d}\mu + \sum_{k=1}^{\infty} \int_{B\cap\{\gamma D_k\le w\le\gamma D_{k+1}\}} w^{p-1} w \,\mathrm{d}\mu \\ &\le \gamma^p D_1^p \mu(B) + \sum_{k=1}^{\infty} (\gamma D_{k+1})^{p-1} \int_{B\cap\{w\ge\gamma D_k\}} w \,\mathrm{d}\mu. \end{aligned} \tag{3.7}$$

For $k = 1, 2, \ldots$ we apply Lemma 3.4.2 repeatedly to obtain

$$\int_{B\cap\{w\ge\gamma D_k\}} w \,\mathrm{d}\mu \le \gamma^{-(k-1)\beta} w(2B).$$

Combining this with (3.7), we obtain a series that converges by virtue of the choice of p, and conclude that there exists a constant C depending on the parameters in (ii), as well as on C_d, γ, β, and p, such that

$$\fint_B w^p \,\mathrm{d}\mu \le C \left(\frac{w(2B)}{\mu(2B)} \right)^p .$$

A peculiarity of weak weights is the upper bound $\eta < C_d^{-5}$ that appears in the statement of (ii). This bound tends to zero with increasing dimension and cannot be done away with, unlike in the case of Muckenhoupt weights where any $0 < \eta, \varepsilon < 1$ will do. The same phenomenon occurs already in Spadaro's Euclidean proof [60]. Example 4.2 in Publication III demonstrates that the upper bound must be smaller than 2^{1-n}, which was observed by Sawyer in dimension 2 in [58].

Statement (iii), called the Fujii–Wilson condition [28, 65], has previously been shown for weak A_∞ weights by Anderson, Hytönen, and Tapiola [1]. They prove that (iii) $\Rightarrow$ (ii) by means of weak weights defined on Christ-type dyadic systems of cubes; again see [12]. We show that (iii) $\Rightarrow$ (ii) by elementary arguments. Using nothing but fundamental properties of the maximal function and the Vitali covering lemma, one arrives at the following weak version of (i).

Lemma 3.4.3. *For every $\widetilde{\eta} > 0$ there exists $\widetilde{\varepsilon} > 0$ such that for every measurable set $F \subset B$ with $4B \Subset \Omega$ and $\mu(F) \le \widetilde{\varepsilon}\mu(B)$, we have $w(F) \le \widetilde{\eta} w(4B)$.*

To tighten this statement we need one more covering by balls. Let B be a ball such that $2B \Subset \Omega$, and $F \subset B$ a measurable set. An argument involving the Vitali covering lemma provides a collection of balls $\{B_i\}_{i=1}^N$ with $N = N(C_d)$ and $\mathrm{r}(B_i) = 1/5\,\mathrm{r}(B)$, implying that $\mu(B_i) \ge \widetilde{c}(C_d)\mu(B)$.

For any $\eta > 0$ write $\widetilde{\eta} = N\eta$, and $\varepsilon = \widetilde{c}(C_d)\widetilde{\varepsilon}$. If $\mu(F) \le \varepsilon\mu(B)$, then

$$\frac{\mu(F\cap B_i)}{\mu(B_i)} \le \frac{\mu(F)}{\widetilde{c}(C_d)\mu(B)} \le \widetilde{\varepsilon},$$

which by Lemma 3.4.3 implies that $w(F \cap B_i) \le \tilde{\eta} w(4B_i) \le \tilde{\eta} w(2B)$ for every $i = 1, \dots, N$. We conclude that

$$w(F) \le \sum_{i=1}^{N} w(F \cap B_i) \le \tilde{\eta} \sum_{i=1}^{N} w(4B_i) \le \tilde{\eta} N w(2B) = \eta w(2B).$$

Condition (iv) is a generalization of [58, Formula 23], which was originally stated for Muckenhoupt weights in $\mathbb{R}^n$. It deserves attention for bringing up the well-known connection between A_p and BMO. Namely, whenever $p > 1$, we have

$$\mathrm{BMO}(\Omega) = \{\alpha \log(w) : \alpha \ge 0 \text{ and } w \in A_p(\Omega)\}\ ;$$

see [59, Theorem 3.11] and the remark thereafter. Assuming the weak reverse Hölder inequality (i), statement (iv) essentially follows from the John–Nirenberg inequality. The reverse implication is (iv) $\Rightarrow$ (ii), whose proof involves a weak version of (ii) and a covering argument identical to the above.

3.5 Symmetry of the limiting classes

The case $p = \infty$ is the limiting class of reverse Hölder conditions both weak and strong. For an open set $\Omega \subset X$ we say that the weight w belongs to the class $RH_\infty(\Omega)$, if there exists a constant $C > 0$ such that for every ball $B \Subset \Omega$

$$\operatorname*{ess\,sup}_B w \le C \fint_B w \,\mathrm{d}\mu. \tag{3.8}$$

The corresponding *weak* RH_∞ class is denoted $WRH_\infty(\Omega)$, and involves w_{2B} on the right-hand side for balls B such that $2B \Subset \Omega$. From the discussion in Publications III–IV we gather that

$$RH_\infty \subsetneq \bigcap_{s>1} RH_s\,, \quad WRH_\infty \subsetneq \bigcap_{s>1} WRH_s\,,$$

$$RH_\infty \subsetneq WRH_\infty\,, \quad RH_s \subsetneq WRH_s.$$

We might recall that $A_1 \subsetneq \bigcap_{p>1} A_p$, with an example in [43] showing that the inclusion is proper. Indeed, the relations of A_1 and RH_∞ to A_p and RH_s when $1 < r, s < \infty$, respectively, exhibit a certain symmetry that is all the more apparent in contexts where the RHI characterizes A_p weights. This is the topic of Publication IV (PIV), culminating in Theorems 3.5.2–3.5.3 that provide closely analogous characterizations of A_1 and RH_∞ on metric measure spaces.

Previously, Cruz-Uribe–Neugebauer [17] have systematically investigated the structure of reverse Hölder classes in Euclidean spaces. To this end, given a $f \in L^1_{\mathrm{loc}}(X)$, they introduce the minimal function

$$mf(x) = \inf_{B \ni x} \fint_B |f| \, \mathrm{d}\mu, \tag{3.9}$$

where the infimum is taken over all balls $B \subset X$ containing the point x. Needless to say, the minimal function mirrors the classical Hardy–Littlewood maximal function. We are now able to express the A_1 and RH_∞ conditions in terms of the two extremal functions:

$$w \in A_1(X) \quad \text{if and only if} \quad Mw(x) \leq [w]_1 \, w(x) \quad \text{for a.e. } x \in X,$$

$$w \in RH_\infty(X) \quad \text{if and only if} \quad w(x) \leq Cmw(x) \quad \text{for a.e. } x \in X.$$

Ou [53, 54] has studied the *natural minimal* and *maximal functions* of a function $f \in L^1_{\mathrm{loc}}(X)$, respectively

$$M^\natural f(x) = \sup_{B \ni x} \fint_B f \, \mathrm{d}\mu, \quad m^\natural f(x) = \inf_{B \ni x} \fint_B f \, \mathrm{d}\mu,$$

that is, (1.7) and (3.9) without absolute value signs. Since weights are nonnegative by definition, this does not concern us overmuch. In [54] it is shown that the natural extremal functions commute with the logarithm on $A_\infty(X)$. For the maximal function, for instance, it follows from the Jensen and reverse Jensen (3.3) inequalities that

$$0 \leq \left(\log M^\natural - M^\natural \log \right) w \leq \log \, [w]_\infty \, ; \tag{3.10}$$

see PIV, Lemma 3.2. The result for the minimal function is the same. The second observation is that the natural extremal functions may be used to characterize functions of bounded upper and lower oscillation (*BUO* and *BLO*). Respectively, these are spaces of those functions $f \in L^1_{\mathrm{loc}}(X)$ for which

$$\|f\|_{BUO} = \sup_{B \subset X} \left(\operatorname*{ess\,sup}_{x \in B} f(x) - \fint_B f \, \mathrm{d}\mu \right) < \infty,$$

$$\|f\|_{BLO} = \sup_{B \subset X} \left(\fint_B f \, \mathrm{d}\mu - \operatorname*{ess\,inf}_{x \in B} f(x) \right) < \infty.$$

We immediately observe that $\|-f\|_{BUO} = \|f\|_{BLO}$, implying that neither of these functionals is a norm. In fact, *BUO* and *BLO* are subsets of *BMO*. The characterization lemma [2, Lemma 2] states the following.

Lemma 3.5.1. *Let X be a metric space with a doubling measure, and $f \in L^1_{\mathrm{loc}}(X)$. Then $f \in BLO(X)$ if and only if $M^\natural f - f \in L^\infty(X)$, with*

$$\|f\|_{BLO} = \|M^\natural f - f\|_\infty,$$

and $f \in BUO(X)$ if and only if $f - m^\natural f \in L^\infty(X)$, with

$$\|f\|_{BUO} = \|f - m^\natural f\|_\infty.$$

Using (3.10) and Lemma 3.5.1 we obtain the following nearly symmetrical descriptions of A_1 and RH_∞, bringing [54] into metric spaces with a doubling measure.

Theorem 3.5.2. *Let X be a metric space with a doubling measure, and w a weight. Then $w \in A_1(X)$ if and only if $w \in A_\infty(X)$ and $\log w \in BLO(X)$. Furthermore,*

$$\exp\left(\|\log w\|_{BLO}\right) \le [w]_1 \le [w]_\infty \exp\left(\|\log w\|_{BLO}\right).$$

Theorem 3.5.3. *Let X be a metric space with a doubling measure and satisfying an annular decay property, and w a weight. Then $w \in A_\infty(X) \cap RH_\infty(X)$ if and only if $\log w \in BUO(X)$ with*

$$C \le \exp(\|\log w\|_{BUO}) \le C\,[w]_\infty\,,$$

where C is the infimal constant in the RH_∞ condition (3.8) for w.

Writing $w \in e^{BLO}$ whenever $\log w \in BLO$ etc., we can summarise these results in a single line:

$$A_1 = A_\infty \cap e^{BLO}, \quad RH_\infty = e^{BUO}.$$

In the proof of Theorem 3.5.3, annular decay or another suitable assumption is needed in order to connect A_p and reverse Hölder information. The statement of the theorem is in fact redundant, because

$$RH_\infty \subsetneq \bigcap_{s>1} RH_s \subset \bigcup_{s>1} RH_s = A_\infty.$$

In other words, $w \in RH_\infty$ implies that $w \in A_\infty$. The first inclusion is [17, Theorem 4.1] (in $\mathbb{R}^n$, but the proof in metric spaces is the same). The final equality is valid under the annular decay assumption; see [47] and the discussion above.

Using the characterizations provided by Theorems 3.5.2–3.5.3, several well known properties of A_p weights can be shown by little more than arithmetic with exponents on the oscillation side. The downside of operating at this level of abstraction is that one tends to lose quantitative information: compare PIV, Lemma 2.6, and Propositions 4.2–4.4 in [54]. The key application of these techniques in Publication IV is the following refined Jones factorization theorem that incorporates both A_p and reverse Hölder data.

Theorem 3.5.4. *Let X be a metric space with a doubling measure and satisfying an annular decay property, and $1 < p, s < \infty$. The weight $w \in A_p(X) \cap RH_s(X)$ if and only if $w = w_1 w_2$ for some $w_1 \in A_1(X) \cap RH_s(X)$ and $w_2 \in A_p(X) \cap RH_\infty(X)$.*

Furthermore, Publication IV generalizes the following boundedness result for the natural maximal function, shown in [53] on Euclidean spaces.

Proposition 3.5.5. *Let X be a metric space with a doubling measure and satisfying an annular decay property. If $f \in BMO(X)$, then $M^\natural f \in BLO(X)$ with*

$$\|M^\natural f\|_{BLO} \leq C(C_d)\|f\|_{BMO}.$$

The proof requires both the Coifman–Rochberg (PII, Proposition 2.10) and John–Nirenberg lemmas. The Coifman–Rochberg lemma ensures that M maps A_∞ to A_1. One might conjecture that the corresponding boundedness result for the minimal function would involve RH_∞ in lieu of A_1.

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Aalto-DT 151/2022

ISBN 978-952-64-0985-6 (printed)
ISBN 978-952-64-0986-3 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)

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